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**Further mathematics**  
**Higher level**  
**Paper 1**

Friday 23 October 2020 (afternoon)

2 hours 30 minutes

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 6]

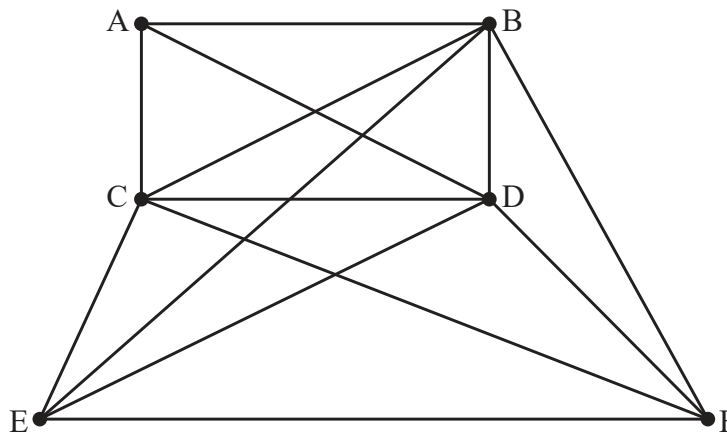
Use l'Hôpital's rule to determine the value of

$$\lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} . \quad [6]$$

2. [Maximum mark: 8]

The following diagram shows the graph  $G$ .

diagram not to scale



- (a) Verify that  $G$  satisfies the handshaking lemma. [3]
- (b) Show that  $G$  cannot be redrawn as a planar graph. [3]
- (c) State, giving a reason, whether  $G$  contains an Eulerian circuit. [2]

3. [Maximum mark: 8]

The binary operation  $*$  is defined on the set  $S = \{a, b, c, d, e, f\}$  by the following Cayley table.

$*$	$a$	$b$	$c$	$d$	$e$	$f$
$a$	$c$	$e$	$a$	$f$	$d$	$b$
$b$	$d$	$c$	$b$	$e$	$f$	$a$
$c$	$a$	$b$	$c$	$d$	$e$	$f$
$d$	$b$	$f$	$d$	$c$	$a$	$e$
$e$	$f$	$a$	$e$	$b$	$c$	$d$
$f$	$e$	$d$	$f$	$a$	$b$	$c$

- (a) Explain why this table is a Latin square. [1]
- (b) State the identity element. [1]
- (c) Determine the inverse of each element of  $S$ . [1]
- (d) Find
  - (i)  $a * (b * d)$ ;
  - (ii)  $(a * b) * d$ . [3]
- (e) State, giving a reason, whether  $\{S, *\}$  is a group. [2]

4. [Maximum mark: 11]

The matrix  $A$  is given by  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

- (a) By considering the determinant of a relevant matrix, show that the eigenvalues,  $\lambda$ , of  $A$  satisfy the equation

$$\lambda^2 - \alpha\lambda + \beta = 0,$$

where  $\alpha$  and  $\beta$  are functions of  $a, b, c, d$  to be determined. [4]

- (b) (i) Verify that

$$A^2 - \alpha A + \beta I = 0.$$

- (ii) Assuming that  $A$  is non-singular, use the result in part (b)(i) to show that

$$A^{-1} = \frac{1}{\beta}(\alpha I - A). \quad [7]$$

Turn over

5. [Maximum mark: 8]

The continuous random variable  $X$  has cumulative distribution function  $F$ , where  $F(a) = 0$  and  $F(b) = 1$ .

(a) Using integration by parts, show that  $E(X) = b - \int_a^b F(x)dx$ . [4]

$$\text{Let } F(x) = \begin{cases} 0, & x < 0 \\ \tan x, & 0 \leq x \leq \frac{\pi}{4} \\ 1, & x > \frac{\pi}{4} \end{cases} .$$

(b) Using the result from part (a), determine  $E(X)$ . Give your answer correct to three significant figures. [2]

(c) Determine the median of  $X$ , giving your answer correct to three significant figures. [2]

6. [Maximum mark: 6]

Find the smallest positive value of  $x$  satisfying the following two linear congruences simultaneously.

$$\begin{aligned} 5x &\equiv 4 \pmod{11} \\ 11x &\equiv 6 \pmod{7} \end{aligned} \quad [6]$$

7. [Maximum mark: 12]

Points in the plane are subjected to a transformation  $T$  in which the point  $(x, y)$  is transformed to the point  $(x', y')$  where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

(a) Describe, in words, the effect of the transformation  $T$ . [1]

(b) (i) Show that the points  $A(1, 4), B(4, 8), C(8, 5), D(5, 1)$  form a square.

(ii) Determine the area of this square.

(iii) Find the coordinates of  $A', B', C', D'$ , the points to which  $A, B, C, D$  are transformed under  $T$ .

(iv) Show that  $A' B' C' D'$  is a parallelogram.

(v) Determine the area of this parallelogram. [11]

8. [Maximum mark: 12]

Consider the group  $\{S, \times_{13}\}$ , where  $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  and  $\times_{13}$  denotes multiplication modulo 13.

- (a) Find the five pairs of distinct elements of  $S$ , such that each element in a pair is the inverse of the other element in the pair. [4]
- (b) Determine the subgroup of  $\{S, \times_{13}\}$ 
  - (i) of order 2;
  - (ii) of order 3. [3]
- (c) You are given that  $\{T, \times_{13}\}$  is a subgroup of  $\{S, \times_{13}\}$ , where  $T = \{1, 5, 8, 12\}$ .
  - (i) Determine the cosets of the elements 2, 3 and 4 with respect to  $\{T, \times_{13}\}$ .
  - (ii) State the general result concerning the elements contained in different cosets that is verified by your answer to part (c)(i). [5]

9. [Maximum mark: 13]

The discrete random variable  $X$  has probability distribution

$$P(X=x) = pq^x, x \in \mathbb{N}, 0 < p < 1, q = 1 - p.$$

- (a) (i) Show that the probability generating function of  $X$  is given by

$$G_x(t) = \frac{p}{1-qt}.$$

- (ii) Hence find  $\text{Var}(X)$  in terms of  $p$ . Express your answer in its simplest form. [9]
- (b) The random variable  $Y$  is defined by

$$Y = X_1 + X_2 + X_3 + X_4$$

where  $X_1, X_2, X_3, X_4$  is a random sample from the distribution of  $X$ .

- (i) Write down the probability generating function of  $Y$ .
- (ii) Hence determine an expression for  $P(Y=3)$  in terms of  $p$ . [4]

Turn over

10. [Maximum mark: 7]

The matrix  $M$  is given by

$$M = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 2 & 3 \\ -1 & 4 & 0 & 5 \\ 1 & 7 & 1 & 9 \end{bmatrix}.$$

(a) Justifying your answer, determine the rank of  $M$ . [3]

Let the set  $S = \left\{ \begin{bmatrix} 2 \\ 5 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \\ 9 \end{bmatrix} \right\}$ , that is the four columns of  $M$ .

(b) Give a reason why  $S$  does not span the space of four-dimensional column vectors. [1]

(c) Determine whether the vector  $\begin{bmatrix} 7 \\ 12 \\ 2 \\ 9 \end{bmatrix}$  belongs to the subspace spanned by  $S$ . [3]

11. [Maximum mark: 12]

(a) Use the integral test to show that the infinite series

$$S = \sum_{n=2}^{\infty} \frac{\ln n}{n^2} \text{ is convergent.} \quad [8]$$

(b) (i) Sketch the graph of  $y = \frac{\ln x}{x^2}$  for  $x \geq 2$ .

(ii) Hence by considering appropriate Riemann sums, show that an upper bound for  $S$  is  $\frac{1}{2} + \frac{3}{4} \ln 2$ . [4]

12. [Maximum mark: 6]

The points D, E, F lie on the sides [BC], [CA], [AB], respectively, of a triangle ABC. The segments [AD], [BE], [CF] meet at O. Given that [FE] is parallel to [BC], show that  $BD = CD$ .

[6]

13. [Maximum mark: 10]

Observations on 12 pairs of values of the random variables  $X, Y$  yielded the following results.

$$\Sigma x = 76.3, \Sigma x^2 = 563.7, \Sigma y = 72.2, \Sigma y^2 = 460.1, \Sigma xy = 495.4$$

(a) (i) Calculate the value of  $r$ , the product moment correlation coefficient of the sample.

(ii) Assuming that the distribution of  $X, Y$  is bivariate normal with product moment correlation coefficient  $\rho$ , calculate the  $p$ -value of your result when testing the hypotheses  $H_0: \rho = 0; H_1: \rho > 0$ .

(iii) State whether your  $p$ -value suggests that  $X$  and  $Y$  are independent.

[7]

(b) Given a further value  $x = 5.2$  from the distribution of  $X, Y$ , predict the corresponding value of  $y$ . Give your answer to one decimal place.

[3]

14. [Maximum mark: 11]

In the triangle ABC,  $AB = 8, BC = 12$  and  $AC = 10$ . A circle is inscribed in this triangle.

(a) Find the lengths of the tangents from A, B and C to this inscribed circle.

[3]

(b) (i) Show that the area of the triangle ABC is  $15r$ , where  $r$  denotes the radius of the inscribed circle.

(ii) Show that  $\sin \hat{A} = \frac{3\sqrt{7}}{8}$ .

(iii) Using parts (b) (i) and (ii), or otherwise, show that  $r$  is equal to  $\sqrt{N}$ , where  $N$  is a positive integer whose value is to be determined.

[8]

15. [Maximum mark: 7]

Let  $(1021)_n$  denote a number expressed in number base  $n$ .

Use mathematical induction to prove that  $(1021)_n$  is not divisible by 3, for  $n \geq 3$ .

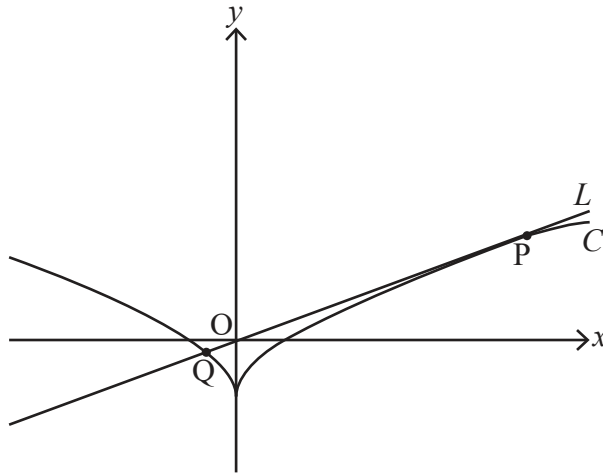
[7]

Turn over



16. [Maximum mark: 13]

The following diagram shows part of the curve  $C$  with parametric equations  $x = t^3$ ,  $y = t^2 - 1$ ,  $t \in \mathbb{R}$ .



The line  $L$  passes through the origin  $O$  and is tangential to  $C$  at the point  $P(p^3, p^2 - 1)$ , where  $p > 0$ . The line  $L$  intersects  $C$  again at the point  $Q$ .

(a) Determine

(i) the equation of  $L$ , giving the gradient in its exact form.

(ii) the exact coordinates of  $P$ .

[8]

(b) Determine the exact coordinates of  $Q$ .

[5]